Problem 1

Draw rough sketches of the solution to the IBVP (3.6) for different values of time. Do your sketches satisfy the BCs? What is the steady-state temperature of the rod? Is this obvious based on your intuition?

Solution

The IBVP in equation (3.6) is

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} \quad 0 < x < 200 & 0 < t < \infty \\ \text{BCs} & \begin{cases} u_x(0,t) = 0 \\ u_x(200,t) = -\dfrac{h}{k}[u(200,t) - 20] \end{cases} & 0 < t < \infty. \\ \text{IC} & u(x,0) = 0^\circ\text{C} \quad 0 \le x \le 200 & \end{array}$$

To solve this IBVP, take advantage of the fact that the PDE and its associated conditions are linear. Assume that the temperature has a steady component and an unsteady component.

$$u(x,t) = v(x) + w(x,t)$$

Plug this into the PDE

$$\frac{\partial}{\partial t}[v(x) + w(x,t)] = \alpha^2 \frac{\partial^2}{\partial x^2}[v(x) + w(x,t)] \quad \rightarrow \quad w_t = \alpha^2[v''(x) + w_{xx}]$$

and each of the conditions.

$$\begin{aligned} u_x(0,t) &= 0 & &\rightarrow & v'(0) + w_x(0,t) &= 0 \\ u_x(200,t) &= -\frac{h}{k}[u(200,t) - 20] & &\rightarrow & v'(200) + w_x(200,t) &= -\frac{h}{k}[v(200) + w(200,t) - 20] \\ u(x,0) &= 0 & &\rightarrow & v(x) + w(x,0) &= 0 \end{aligned}$$

To make the PDE and boundary conditions homogeneous for $w(x,t)$, let $v''(x) = 0$ and $v'(0) = 0$ and $v'(200) = -(h/k)[v(200) - 20]$. Then the IBVP for $w(x,t)$ is

$$\begin{aligned} w_t &= \alpha^2 w_{xx} \\ w_x(0,t) &= 0 \\ w_x(200,t) &= -\frac{h}{k} w(200,t) \\ w(x,0) &= -v(x). \end{aligned}$$

Start by solving the ODE for v. The general solution for $v''(x) = 0$ is

$$v(x) = C_1 x + C_2.$$

Take the derivative with respect to x.

$$v'(x) = C_1$$

Apply the boundary conditions to determine C_1 and C_2.

$$v'(0) = C_1 = 0$$

$$v'(200) = C_1 = -\frac{h}{k}[C_1(200) + C_2 - 20] = -\frac{h}{k}[v(200) - 20]$$

Solve this system of equations.

$$C_1 = 0 \qquad C_2 = 20$$

The steady-state temperature is therefore

$$v(x) = 20.$$

Using the method of separation of variables, the solution to the IBVP,

$$w_t = \alpha^2 w_{xx}$$

$$w_x(0,t) = 0$$

$$w_x(200,t) = -\frac{h}{k}w(200,t)$$

$$w(x,0) = -20$$

is found to be

$$w(x,t) = -\sum_{n=1}^{\infty} \frac{80\sin(200\gamma_n)}{400\gamma_n + \sin(400\gamma_n)} e^{-\alpha^2\gamma_n^2 t}\cos(\gamma_n x),$$

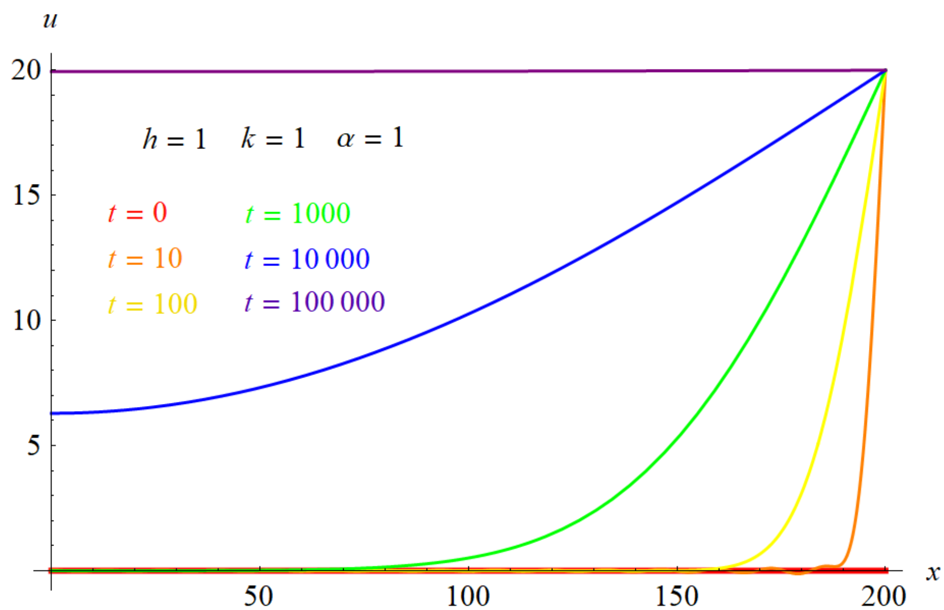
where γ_n is the nth solution to the transcendental equation,

$$\tan(200\gamma_n) = \frac{h}{k\gamma_n}.$$

Therefore, since $u(x,t) = v(x) + w(x,t)$,

$$u(x,t) = 20 - 80\sum_{n=1}^{\infty} \frac{\sin(200\gamma_n)}{400\gamma_n + \sin(400\gamma_n)} e^{-\alpha^2\gamma_n^2 t}\cos(\gamma_n x).$$

Plot this function versus x at several times to illustrate the behavior of this solution.



All curves have a slope of zero at $x = 0$, representing the thermally insulated boundary. Also, the temperature throughout the rod increases due to the ambient fluid temperature of 20°C at $x = 200$. At $t = 0$ the temperature in the rod is uniformly zero, and in the steady state, after a long time has passed, the temperature in the rod is uniformly 20°C, the same as the ambient temperature.